JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 15, No. 3, May-June 1992

Unified Approach to Inertial Navigation System Error Modeling

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Several inertial navigation system error models have been developed and used in the literature. Most of the models are ad hoc models which were needed to solve certain particular problems and were developed for that purpose only. Consequently, the relationship, correspondence, and equivalence between the various models is not evident. This paper presents a new methodology for developing inertial navigation systems error models which also puts all of the known models in the same framework and shows the equivalence between them. The new methodology is based on several choices the developer has to make which uniquely define the error model. This new approach enables the development of all existing models in a unified way, hence the equivalence and correspondence between them is obvious. Moreover, any new model which is of interest can be developed using the methodology presented in this work. In fact, any new model which will ever be developed for the class of systems considered here will fit into the framework described in this paper.

Nomenclature

$\boldsymbol{a}$ = general vector (abstract representation of a general vector)
a_b = column matrix whose entries are the components of the general vector $\boldsymbol{a}$ resolved in a general coordinate system b (realization of $\boldsymbol{a}$ in system b)
C_b^a = transformation matrix from a general coordinate system a to a general coordinate system b
$\boldsymbol{f}$ = specific force
$\boldsymbol{g}$ = gravity vector
$\boldsymbol{g}^m$ = gravitation vector
$\boldsymbol{R}$ = position vector
$\boldsymbol{U}$ = inertially referenced velocity
$\boldsymbol{V}$ = Earth referenced velocity
$\boldsymbol{\gamma}$ = small angle, treated as a vector, from a known coordinate system k to an unknown coordinate system k′
δ = perturbation
$\delta \boldsymbol{n}$ = position error vector perturbed in the c coordinates
δ_q = perturbation in the general q coordinates
$\delta\boldsymbol{\theta}$ = small angle vector from the t coordinate to the c coordinate frame
$\boldsymbol{\epsilon}$ = gyro error vector
$\boldsymbol{\phi}$ = small angle vector from the t coordinate to the p coordinate frame
$\boldsymbol{\psi}$ = small angle vector from the c coordinate to the p coordinate frame

$\omega_q^{ab} = \begin{bmatrix} \omega_{xq} \\ \omega_{yq} \\ \omega_{zq} \end{bmatrix}$ column matrix of the angular velocity vector at which the general coordinate system a rotates with respect to the general coordinate system b, resolved in the general coordinate system q

$\Omega_q^{ab} = \begin{bmatrix} 0 & -\omega_{zq} & \omega_{yq} \\ \omega_{zq} & 0 & -\omega_{xq} \\ -\omega_{yq} & \omega_{xq} & 0 \end{bmatrix}$ cross product matrix of ω_q^{ab}

∇ = accelerometer error vector

Notations

$(\,)_q$ = column matrix whose entries from top to bottom are the x, y, and z elements of ( ) when resolved in the general coordinate system q [realization of ( ) in system q]
$\overset{q}{(\,)}$ = time derivative of vector ( ) relative to the general reference coordinates q; that is, the time derivative of ( ) as seen by an observer in the general coordinate frame q

$[(\,)_q \times\,] = \begin{bmatrix} 0 & -(\,)_{zq} & (\,)_{yq} \\ (\,)_{zq} & 0 & -(\,)_{xq} \\ -(\,)_{yq} & (\,)_{xq} & 0 \end{bmatrix}$ cross product matrix of $(\,)_q$

Coordinate Systems

b = body
b* = assumed body
c = computer
e = Earth fixed
i = inertially fixed
p = platform (sensor) fixed
q = general coordinate system
t = true (the true reference coordinates)

I. Introduction

MUCH of the work performed in the field of inertial navigation systems (INS) concentrates around error analysis. This is a direct consequence of the importance of error analysis in the design and operation of INS. This importance becomes obvious after one realizes that the navigation errors determine the performance of the vehicle carrying the INS, and that for a given mission accuracy, the error analysis enables the determination of the required INS sensor accuracy.

Error analysis is based, of course, on error models. The models, besides being the tool for INS error analysis described above, also serve for real-time failure detection and for the implementation of a Kalman filter in the INS. Of the last two

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Received Oct. 31, 1990; revision received May 22, 1991; accepted for publication May 23, 1991.

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purpose, the implementation of a Kalman filter is the most important one since any modern INS utilizes some kind of a Kalman filter for its initial alignment and calibration, and for its update during its operation.

Error models are developed by perturbing the nominal differential equations whose solution yields the INS output; namely, position, velocity, and orientation. The nominal equations are based on the fundamental law of mechanics:

$$\overset{\text{ii}}{\ddot{\boldsymbol{R}}} = \boldsymbol{f} + \boldsymbol{g}^m \tag{1}$$

where $\overset{\text{ii}}{\ddot{\boldsymbol{R}}}$ is the second time derivative of $\boldsymbol{R}$ as seen by an observer in the inertial coordinate frame. However, this basic equation can be expressed in different rotating coordinate systems and thus take different but equivalent forms. This is just one reason why equivalent error models may have different forms. Presently there exist different INS error models.[1-5] The connection between those models is unclear; they normally use different terminology, and the equivalence between most of them is not recognized. There is no unified and systematic methodology for developing error models and each model seems to be a result of a unique approach.

The purpose of this work is to present a unified approach to the development of INS error models. The presentation of the unified approach has two goals. The first is to supply the developer of an INS error model with a tool which clearly indicates the steps that one has to follow, and which presents the variety of choices the developer has. The second goal is to put all INS error models in the same context such that their equivalence can be evident.

At the first step of the development of error models, one has to choose several items such as nominal as well as error variables, coordinate systems, etc. Based on this choice, the developer then applies the suitable perturbations which yield the INS error model.

In the next section, we present the rules for perturbing the INS nominal equations. In Sec. III, we list the choices which have to be made before perturbing the nominal equations, then, in Sec. IV, we present examples of error models, two of which are well known. Three examples demonstrate the unified approach to the development of INS error models and show that the error models in these examples are all equivalent. Finally, in Sec. V, we present the conclusions derived from this work.

II. Perturbation Rules

A particular INS error model is derived from a particular set of INS nominal equations by perturbation. We distinguish between angular and translatory perturbations. We need to use angular perturbations in order to express translatory perturbations; therefore, we start our discussion with the former.

A. Angular Perturbation Rules

Let k denote a certain known coordinate sytsem, and let k′ denote a coordinate system which is supposed to be k but is rather rotated with respect to k by the small unknown angular vector γ. Thus, while k is known, k′ is an unknown coordinate system. We can write

$$\frac{\mathrm{d}C^{\mathrm{k}}_{\mathrm{k}'}}{\mathrm{d}t} = \Omega^{\mathrm{kk}'}_{\mathrm{k}'} C^{\mathrm{k}}_{\mathrm{k}'} \tag{2}$$

Postmultiply Eq. (2) by $C^{\mathrm{k}'}_{\mathrm{k}}\,\mathrm{d}t$ to obtain

$$\mathrm{d}C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{k}'}_{\mathrm{k}} = \Omega^{\mathrm{kk}'}_{\mathrm{k}'}\,\mathrm{d}t \tag{3}$$

It follows that

$$\mathrm{d}C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{k}'}_{\mathrm{k}} = [\omega^{\mathrm{kk}'}_{\mathrm{k}'}\,\mathrm{d}t\,\times\,] \tag{4}$$

We can consider $\boldsymbol{\omega}^{\mathrm{kk}'}_{\mathrm{k}'}$ as the angular rate vector at which system k′ departs from system k; that is, $\omega^{\mathrm{kk}'}_{\mathrm{k}'}$ is the cause of the build up of γ; thus, for a small angular deviation γ, we may write

$$\omega^{\mathrm{kk}'}_{\mathrm{k}'}\,\mathrm{d}t = \gamma_{\mathrm{k}} \tag{5}$$

Consequently,

$$[\omega^{\mathrm{kk}'}_{\mathrm{k}'}\,\mathrm{d}t\,\times\,] = [\gamma_{\mathrm{k}}\times\,] \tag{6}$$

and Eq. (4) becomes

$$\mathrm{d}C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{k}'}_{\mathrm{k}} = [\gamma_{\mathrm{k}}\times\,] \tag{7}$$

We consider $\mathrm{d}C^{\mathrm{k}}_{\mathrm{k}'}$ to be the perturbation on $C^{\mathrm{k}}_{\mathrm{k}'}$ and we denote it by $\delta C^{\mathrm{k}}_{\mathrm{k}'}$; hence Eq. (7) is written as

$$\delta C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{k}'}_{\mathrm{k}} = [\gamma_{\mathrm{k}}\times\,] \tag{8}$$

Consider the following transformation from the known inertial coordinate system through the known system k to the unknown system k′:

$$C^{\mathrm{i}}_{\mathrm{k}'} = C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{i}}_{\mathrm{k}} \tag{9}$$

Perturbation of Eq. (9) yields

$$\delta C^{\mathrm{i}}_{\mathrm{k}'} = \delta C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{i}}_{\mathrm{k}} + C^{\mathrm{k}}_{\mathrm{k}'} \delta C^{\mathrm{i}}_{\mathrm{k}} \tag{10}$$

However, perturbation of the transformation matrix between known coordinate systems yields zero, thus

$$\delta C^{\mathrm{i}}_{\mathrm{k}} = 0 \tag{11}$$

Postmultiply the remainder of Eq. (10) by $C^{\mathrm{k}'}_{\mathrm{i}}$ to obtain

$$\delta C^{\mathrm{i}}_{\mathrm{k}'} C^{\mathrm{k}'}_{\mathrm{i}} = \delta C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{k}'}_{\mathrm{k}} \tag{12}$$

Substitution of Eq. (8) into the right-hand side of Eq. (12) yields

$$\delta C^{\mathrm{i}}_{\mathrm{k}'} C^{\mathrm{k}'}_{\mathrm{i}} = [\gamma_{\mathrm{k}}\times\,] \tag{13}$$

B. Translatory Perturbation Rules

Let q be any known coordinate system, and let $\boldsymbol{a}$ be any vector, then perturbation of the following obvious relation

$$\boldsymbol{a}_{\mathrm{k}'} = C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{q}}_{\mathrm{k}} \boldsymbol{a}_{\mathrm{q}} \tag{14}$$

[where k and k′ are the coordinate systems specified in Eq. (2)] yields

$$\delta\boldsymbol{a}_{\mathrm{k}'} = C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{q}}_{\mathrm{k}} \delta\boldsymbol{a}_{\mathrm{q}} + \delta C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{q}}_{\mathrm{k}} \boldsymbol{a}_{\mathrm{q}} \tag{15}$$

Using Eq. (8) in Eq. (15) and using the relation $C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{q}}_{\mathrm{k}} = C^{\mathrm{q}}_{\mathrm{k}'}$ yields

$$\delta\boldsymbol{a}_{\mathrm{k}'} = C^{\mathrm{q}}_{\mathrm{k}'} \delta\boldsymbol{a}_{\mathrm{q}} + [\gamma_{\mathrm{k}}\times\,] C^{\mathrm{k}}_{\mathrm{k}'} C^{\mathrm{q}}_{\mathrm{k}} \boldsymbol{a}_{\mathrm{q}} \tag{16}$$

or

$$\delta\boldsymbol{a}_{\mathrm{k}'} = C^{\mathrm{q}}_{\mathrm{k}'} \delta\boldsymbol{a}_{\mathrm{q}} + [\gamma_{\mathrm{k}}\times\,] \boldsymbol{a}_{\mathrm{k}'} \tag{17}$$

Because of the nature of the small angular difference vector γ, we can replace γ_{k} in Eq. (17) by $\gamma_{\mathrm{k}'}$; thus,

$$\delta\boldsymbol{a}_{\mathrm{k}'} = C^{\mathrm{q}}_{\mathrm{k}'} \delta\boldsymbol{a}_{\mathrm{q}} + [\gamma_{\mathrm{k}'}\times\,] \boldsymbol{a}_{\mathrm{k}'} \tag{18}$$

The last equation can be transformed from its matrix form to the following vector form

$$\delta_{\mathrm{k}'}\boldsymbol{a} = \delta_{\mathrm{q}}\boldsymbol{a} + \gamma \times \boldsymbol{a} \tag{19}$$

We say that $\delta_{k'}\boldsymbol{a}$ is the error in $\boldsymbol{a}$ which is apparent to an observer in the k′ coordinates, whereas $\delta_q\boldsymbol{a}$ is the error in $\boldsymbol{a}$ which is apparent to an observer in the q coordinates. Obviously, if both q and k′ are known, then the apparent perturbations in both systems are identical since $\gamma = 0$. A special case of the above is when q = k. For the special case where q = k and $\boldsymbol{a} = \omega^{k'k}$, we obtain

$$\delta_{k'}\omega^{k'k} = \delta_k\omega^{k'k} + \gamma \times \omega^{k'k} \tag{20}$$

However, from Eq. (5), it is seen that $\omega^{k'k}$ and γ are collinear and thus their cross product vanishes; consequently, Eq. (20) becomes

$$\delta_{k'}\omega^{k'k} = \delta_k\omega^{k'k} \tag{21}$$

III. Choices to be Made in INS Error Model Development

The differences between the various INS error models stem from the variety of ways to express the same physical entities. Therefore, the developer has to decide what are the choices most suitable for the problem on hand. The choices are of the following items:

1) Nominal equations. a) Translatory velocity variables, b) translatory equations, and c) reference coordinate system.

2) Error equations. a) Coordinate system for linearization, b) position-error variables, c) velocity-error variables, and d) attitude-error variables.

These choices are discussed in detail next.

A. Nominal Equations

In general, there are two nominal vector differential equations which have to be solved in order to yield the INS desired output. They are known as the translatory equations.

1. Translatory Velocity Variables

The translatory velocity variables used in INS are $\boldsymbol{U} = \overset{i}{\boldsymbol{R}}$ and

$$\boldsymbol{V} = \overset{e}{\boldsymbol{R}}$$

(Mathematically, one can use the position time derivative with respect to other coordinates. However, no known error model uses a velocity other than $\boldsymbol{U}$ or $\boldsymbol{V}$).

2. Translatory Equations

As stated in the Introduction, the basic law of mechanics is

$$\overset{ii}{\boldsymbol{R}} = \boldsymbol{f} + \boldsymbol{g}^m \tag{22}$$

The translatory equations are derived from this basic equation. In other words, the nominal translatory equations are the expression of this basic law in reference rotating coordinates (unless the reference coordinates are the inertial ones). When deriving the translatory equations we are faced with the following choices: 1) use inertial-referenced velocity

$$\boldsymbol{U} = \overset{i}{\boldsymbol{R}}$$

or Earth-referenced velocity

$$\boldsymbol{V} = \overset{e}{\boldsymbol{R}}$$

and 2) use position-based equations or velocity-based equations. We are faced then with the four possibilities summarized in Table 1, where $\boldsymbol{R}$-$\boldsymbol{U}$ corresponds to

$$\overset{qq}{\boldsymbol{R}} + 2\omega^{qi} \times \overset{q}{\boldsymbol{R}} + \overset{q}{\mathrm{w}}{}^{qi} \times \boldsymbol{R} + \omega^{qi} \times (\omega^{qi} \times \boldsymbol{R}) = \boldsymbol{f} + \boldsymbol{g}^m \tag{23a}$$

$$\boldsymbol{U} = \overset{q}{\boldsymbol{R}} + \omega^{qi} \times \boldsymbol{R} \tag{23b}$$

(where $\overset{qq}{\boldsymbol{R}}$ is the second time derivative of $\boldsymbol{R}$ as seen by an observer in the q coordinate frame) $\boldsymbol{R}$-$\boldsymbol{V}$ corresponds to

$$\overset{qq}{\boldsymbol{R}} + 2\omega^{qi} \times \overset{q}{\boldsymbol{R}} + \overset{q}{\omega}{}^{qi} \times \boldsymbol{R} + \omega^{qi} \times (\omega^{qi} \times \boldsymbol{R}) = \boldsymbol{f} + \boldsymbol{g}^m \tag{24a}$$

$$\boldsymbol{V} = \overset{q}{\boldsymbol{R}} + \omega^{qe} \times \boldsymbol{R} \tag{24b}$$

$\overset{q}{\boldsymbol{R}}$-$\boldsymbol{U}$ corresponds to

$$\overset{q}{\boldsymbol{R}} = \boldsymbol{U} - \omega^{qe} \times \boldsymbol{R} \tag{25a}$$

$$\overset{q}{\boldsymbol{U}} = -\omega^{qi} \times \boldsymbol{U} + \boldsymbol{f} + \boldsymbol{g}^m \tag{25b}$$

$\overset{q}{\boldsymbol{R}}$-$\boldsymbol{V}$ corresponds to

$$\overset{q}{\boldsymbol{R}} = \boldsymbol{V} - \omega^{qe} \times \boldsymbol{R} \tag{26a}$$

$$\overset{q}{\boldsymbol{V}} = -(\omega^{qi} + \omega^{ei}) \times \boldsymbol{V} + \boldsymbol{f} + \boldsymbol{g} \tag{26b}$$

and

$$\boldsymbol{g} = \boldsymbol{g}^m - \omega^{ei} \times (\omega^{ei} \times \boldsymbol{R}) \tag{26c}$$

Most of the error models which are used in the literature are based on the nominal Eqs. (24) or (26).

3. Reference Coordinate System

In the presentation of the nominal translatory equations in Sec. III.A.1, we used the general nominal coordinate system q in which the translatory equations are solved. A choice has to be made as to what this nominal coordinate system is. Coordinates which are widely used are local-level north-pointing, wander azimuth, tangent plane, and inertial coordinates.

B. Error Equations

The error model is derived by perturbing the chosen nominal equations. Here, too, several choices have to be made before one can perturb the nominal equations.

1. Coordinate System for Linearization

When developing the error equations, one has to choose the coordinate system in which to perturb the nominal equations in order to arrive at the error model. This coordinate system may be entirely different from the coordinate system in which the nominal equations are expressed and is determined, among other factors, by the INS error sources. Table 2 lists the coordinate systems which were used in the literature[1-6] to express the error equations. The e system is an Earth-fixed system whose origin is at the center of the Earth, its x axis points at the north pole, its y axis points at the Greenwich meridian in the equatorial plane, and the z axis completes the

Table 1 Choices of translatory equations

	Inertial-referenced velocity	Earth-referenced velocity
Position-based equations	$\boldsymbol{R}$-$\boldsymbol{U}$	$\boldsymbol{R}$-$\boldsymbol{V}$
Velocity-based equations	$\overset{q}{\boldsymbol{R}}$-$\boldsymbol{U}$	$\overset{q}{\boldsymbol{R}}$-$\boldsymbol{V}$

Table 2 Coordinate systems for INS error modeling

	Known coordinates	Unknown coordinates
Correct coordinates	e—Earth i—inertial	t—true b—body
Erroneous coordinates	c—computer b*—assumed body	p—platform

system to a right-hand coordinate system. The i system is the well-known inertial system, the t system is the correct nominal coordinate system (chosen in Sec. IIIA.3) at the true position of the vehicle carrying the INS, the b system is a set of coordinates fixed in the vehicle carrying the INS, the c system is the correct nominal coordinate system at the INS-computed position, the b* system is the computed body coordinate system[6] and p is the coordinate system attached to the inertial platform when the INS is a gimbaled one. When the INS is a strapdown system, p is the coordinate system attached to the imaginary so-called "mathematical platform."

As shown in Table 2, the coordinate systems are divided into correct and erroneous coordinates on one hand and into known and unknown (to the INS computer) coordinates on the other hand. The most important distinction is between known and unknown coordinates rather than between correct and erroneous ones. As explained in Sec. II, the reasons for this is that there is no need to perturb the transformation matrix between known coordinates even if they are erroneous. For example, the perturbation of C_c^i, the transformation from the inertial to the computer coordinates, is zero because both coordinate systems are "known" to the INS computer even though c is erroneous. The use of known, albeit erroneous, coordinates whenever possible is recommended as it yields a simpler error model.

The coordinates in which the INS navigation computer processes data are the c coordinates and, as such, are "known" to the INS computer. The INS navigation computer also "knows" the e, i, and b* coordinates. This is why all these coordinate systems are considered known systems.

Note that in error analysis usually the c coordinate system serves as the reference system. That is, the erroneous coordinate system which is used by the INS computer is the reference system. Therefore, the c system is "known" to the computer and any perturbation in the c system has no angular error component. On the other hand, the true coordinate system is unknown to the c system. The difference between the c and t systems is "unknown" to the INS computer. Therefore, perturbations in the t system do contain an angular error term.

2. *Position-Error Variables*

The most popular position-error variables in terrestrial INS are $\delta_c \boldsymbol{R}$ and $\delta_t \boldsymbol{R}$. The position-error variable $\delta_c \boldsymbol{R}$ is a position perturbation in the known computer coordinates and, as such, is identical to the position-error perturbation in any other known coordinates. (Sometimes $\delta_c \boldsymbol{R}$ is denoted by $\delta \boldsymbol{n}$, e.g., Ref. 4, and we too will use this notation.) When the INS position is updated using Earth-referenced position, such as the global positioning system (GPS), for example, it is convenient to use $\delta \boldsymbol{n}$ for it is identical to $\delta_c \boldsymbol{R}$ and the latter is directly measured by GPS. The identity between $\delta \boldsymbol{n}$ and $\delta_c \boldsymbol{R}$ stems from the fact that both c and e coordinate systems are known. The position error $\delta_t \boldsymbol{R}$ is a position perturbation in the true (nominal) axes which are unknown to the navigation computer. This is the position error obtained when the INS position is updated using a known landmark.

3. *Velocity-Error Variables*

As mentioned earlier, usually the nominal equations make use of either the inertial-referenced velocity $\boldsymbol{U}$ or of the Earth-referenced velocity $\boldsymbol{V}$. Similarly, the velocity-error variables are the perturbation of either $\boldsymbol{U}$ or $\boldsymbol{V}$ in an appropriate coordinate system. The error in $\boldsymbol{U}$ is not widely used in terrestrial-INS error models although Arshal[5] used $\delta_t \boldsymbol{U}$ in his so-called absolute model. On the other hand, the use of the error in $\boldsymbol{V}$ is widely used. For example, $\delta_c \boldsymbol{V}$ is used in the popular ψ-angle model[2,7] and $\delta_t \boldsymbol{V}$ is used in the perturbation model.[7]

4. *Attitude-Error Variables*

Of the coordinate systems listed in Table 2, the most popular and widely used in error analysis are the t, c, and p systems. The attitude differences between them are small, and therefore can be expressed as vectors of small difference angles rather than sequences of Euler angles. The attitude differences between these coordinate systems are the attitude-error variables widely used in INS error models. The attitude difference between the t and the c coordinates is denoted by $\delta\boldsymbol{\theta}$. It is the angular vector by which system t has to be rotated in order to coincide with system c. Similarly $\boldsymbol{\psi}$, which is the difference between the c and p systems, is the angular vector by which system c has to be rotated in order to coincide with system p. Finally, $\boldsymbol{\phi}$ is the difference between the t and p systems and is defined as the angular vector by which system t has to be rotated in order to coincide with system p. Obviously,

$$\boldsymbol{\phi} = \delta\boldsymbol{\theta} + \boldsymbol{\psi} \tag{27}$$

IV. Examples

A. Psi-Angle Model

Probably the most popular INS error model is the psi-angle model.[2,7] This model is the result of the following choices:

1) Nominal equations. a) Translatory velocity variable $\boldsymbol{V}$; b) translatory equations: *velocity-based equations*; and c) reference coordinates: c (*computer*).

2) Error equations. a) Coordinate system for linearization: *c*; b) position-error variable: $\delta\boldsymbol{n}$; c) velocity-error variable: $\delta_c \boldsymbol{V}$; and d) attitude-error variable: $\boldsymbol{\psi}$.

From choices 1a and 1b, the nominal translatory equations are Eqs. (26) and from choice 1c the general coordinate system q is system c; that is

$$\overset{c}{\dot{\boldsymbol{R}}} = \boldsymbol{V} - \boldsymbol{\omega}^{ce} \times \boldsymbol{R} \tag{28a}$$

$$\overset{c}{\dot{\boldsymbol{V}}} = -(\boldsymbol{\omega}^{ci} + \boldsymbol{\omega}^{ei}) \times \boldsymbol{V} + \boldsymbol{f} + \boldsymbol{g} \tag{28b}$$

According to 2a, we perturb these equations in the (known) c coordinates. Making use of the fact that the perturbation of the derivative is equal to the derivative of the perturbation, and using the above choices 2b and 2c, we obtain

$$\delta_c \overset{c}{\dot{\boldsymbol{R}}} = \delta_c \boldsymbol{V} - \delta_c \boldsymbol{\omega}^{ce} \times \boldsymbol{R} - \boldsymbol{\omega}^{ce} \times \delta_c \boldsymbol{R} \tag{29a}$$

$$\delta_c \overset{c}{\dot{\boldsymbol{V}}} = -\delta_c(\boldsymbol{\omega}^{ci} + \boldsymbol{\omega}^{ei}) \times \boldsymbol{V} - (\boldsymbol{\omega}^{ci} + \boldsymbol{\omega}^{ei}) \times \delta_c \boldsymbol{V} + \delta_c \boldsymbol{f} + \delta_c \boldsymbol{g} \tag{29b}$$

Now since c, e, and i are known coordinates, the angular velocity between them is also known and therefore it follows that

$$\delta\boldsymbol{\omega}^{ce} = \delta\boldsymbol{\omega}^{ci} = \delta\boldsymbol{\omega}^{ei} = 0 \tag{30a}$$

$$\delta_c \boldsymbol{R} = \delta\boldsymbol{n} \tag{30b}$$

therefore Eq. (29) can be written as

$$\overset{c}{\delta\dot{\boldsymbol{n}}} = \delta_c \boldsymbol{V} - \boldsymbol{\omega}^{ce} \times \delta\boldsymbol{n} \tag{31a}$$

$$\delta_c \overset{c}{\dot{\boldsymbol{V}}} = -(\boldsymbol{\omega}^{ci} + \boldsymbol{\omega}^{ei}) \times \delta_c \boldsymbol{V} + \delta_c \boldsymbol{f} + \delta_c \boldsymbol{g} \tag{31b}$$

Following choice 2d, we obtain from Eq. (19),

$$\delta_p \boldsymbol{f} = \delta_c \boldsymbol{f} + \boldsymbol{\psi} \times \boldsymbol{f} \tag{32}$$

thus

$$\delta_c \boldsymbol{f} = \delta_p \boldsymbol{f} - \boldsymbol{\psi} \times \boldsymbol{f} \tag{33}$$

The perturbation $\delta_p \boldsymbol{f}$ is the perturbation on the specific force as measured in the platform system. It is precisely the accelerometer measurement errors ∇; therefore

$$\delta_c \boldsymbol{f} = \nabla - \boldsymbol{\psi} \times \boldsymbol{f} \tag{34}$$

We note that ψ is generated physically by the gyro drifts; that is,

$$\overset{i}{\psi} = \epsilon \tag{35}$$

The rule for differentiation in rotating coordinate systems is

$$\overset{i}{\psi} = \overset{c}{\psi} + \omega^{ci} \times \psi \tag{36}$$

Equating (35) and (36) yields

$$\overset{c}{\psi} + \omega^{ci} \times \psi = \epsilon \tag{37}$$

We realize that in developing Eq. (37), we indeed chose to express the attitude error by ψ as indicated in choice 2d. Equations (31), (33), and (37) yield the following psi-angle error model:

$$\delta \overset{c}{n} = \delta_c V - \omega^{ce} \times \delta n \tag{38a}$$

$$\delta \overset{c}{V} = -(\omega^{ci} + \omega^{ei}) \times \delta_c V + \nabla - \psi \times f + \delta_c g \tag{38b}$$

$$\overset{c}{\psi} = -\omega^{ci} \times \psi + \epsilon \tag{38c}$$

Note that Eq. (38a) is identical to Eq. (4.74) of Ref. 2, Eq. (38b) is identical to Eq. (10) in Ref. 7, and Eq. (38c) is identical to Eq. (4.54) in Ref. 2 and Eq. (13) in Ref. 7.

B. Perturbation Error Model

This model too is a well-known model.[4,7] Benson[7] showed the equivalence between this model and the psi-angle model analytically and by simulation. This model is a result of the following choices:

1) Nominal equations. a) Translatory velocity variable: V; b) translatory equations: *velocity-based equations*; and c) reference coordinates: t (*true*).

2) Error equations. a) Coordinate system for linearization: t; b) position-error variable: δn; c) velocity-error variable: $\delta_t V$; and d) attitude-error variable: ϕ.

From choices 1a, 1b, and 1c, the nominal translatory equation are obtained from Eq. (26) as follows:

$$\overset{t}{R} = V - \omega^{te} \times R \tag{39a}$$

$$\overset{t}{V} = -(\omega^{ti} + \omega^{ei}) \times V + f + g \tag{39b}$$

Perturbation of Eq. (39a) in the (unknown) t coordinates according to the perturbation rules, when using the above choices 2a–2c, yields

$$\delta_t \overset{t}{R} = \delta_t V - \omega^{te} \times \delta_t R - \delta_t \omega^{te} \times R \tag{40}$$

The position error in the perturbation model is expressed in terms of $\delta n = \delta_c R$. After some lengthy development, which uses the perturbation rules developed in Sec. II, one can transform Eq. (40) into the following equation (Ref. 8, Appendix A):

$$\delta \overset{t}{n} = \delta_t V - \omega^{te} \times \delta n + \delta\theta \times V \tag{41}$$

Perturbation of Eq. (39b) in t yields

$$\delta_t \overset{t}{V} = -(\delta_t \omega^{ti} + \delta_t \omega^{ei}) \times V - (\omega^{ti} + \omega^{ei}) \times \delta_t V + \delta_t f + \delta_t g \tag{42}$$

To obtain the attitude error equation for ϕ (according to choice 2d), differentiate Eq. (27) to obtain

$$\overset{t}{\phi} = \delta\overset{t}{\theta} + \overset{t}{\psi} \tag{43}$$

The angle ψ is generated by the gyro drift which is the drift of an inertial sensor; therefore,

$$\overset{i}{\psi} = \epsilon \tag{44}$$

On the other hand,

$$\overset{i}{\psi} = \overset{t}{\psi} + \omega^{ti} \times \psi \tag{45}$$

From the last two equations, we obtain

$$\overset{t}{\psi} = -\omega^{ti} \times \psi + \epsilon \tag{46}$$

From Eq. (27),

$$\psi = \phi - \delta\theta \tag{47}$$

We identify $\delta\overset{t}{\theta} + \omega^{ti} \times \delta\theta$ as $\delta_t \omega^{ti}$; therefore, when using it in Eq. (43) together with Eqs. (46) and (47), we obtain

$$\overset{t}{\phi} = -\omega^{ti} \times \phi + \delta_t \omega^{ti} + \epsilon \tag{48}$$

Equations (41), (42), and (48) yield the following perturbation error model:

$$\delta \overset{t}{n} = \delta_t V - \omega^{te} \times \delta n + \delta\theta \times V \tag{49a}$$

$$\delta_t \overset{t}{V} = -(\delta_t \omega^{ti} + \delta_t \omega^{ei}) \times V - (\omega^{ti} + \omega^{ei}) \times \delta_t V + \delta_t f + \delta_t g \tag{49b}$$

$$\overset{t}{\phi} = -\omega^{ti} \times \phi + \delta_t \omega^{ti} + \epsilon \tag{49c}$$

Note that Eq. (49a) is identical to Eq. (8.82) of Ref. 4 and Eq. (50) of Ref. 7, Eq. (49b) corresponds directly to Eq. (22) of Ref. 7, and Eq. (49c) is identical to Eq. (25) of Ref. 7.

C. Absolute Error Model

This model was first introduced by Arshal in 1987.[5] One may arrive at this model by making the following selections:

1) Nominal equations. a) Translatory velocity variable: U; b) translatory equations: *velocity-based equations*; and c) reference coordinates: t.

2) Error equations. a) Coordinate system for linearization: t; b) position-error variable: $\delta_t R$; c) velocity-error variable: $\delta_t U$; and d) attitude-error variable: $\delta\theta$.

From 1a–1c, we obtain the following nominal equations:

$$\overset{t}{R} = U - \omega^{ti} \times R \tag{50a}$$

$$\overset{t}{U} = -\underset{t}{\omega}^{ti} \times U + f + g^m \tag{50b}$$

Perturbation of Eq. (50) employing the choices 2a–2c yields

$$\delta_t \overset{t}{R} = \delta_t U - \delta_t \omega^{ti} \times R - \omega^{ti} \times \delta_t R \tag{51a}$$

$$\delta_t \overset{t}{U} = -\delta_t \omega^{ti} \times U - \omega^{ti} \times \delta U + \delta_t f + \delta_t g^m \tag{51b}$$

To derive the attitude-error equation for $\delta\theta$ (according to choice 2d), we use Eq. (43) as follows:

$$\delta\overset{t}{\theta} = \overset{t}{\phi} - \overset{t}{\psi} \tag{52}$$

Now

$$\overset{t}{\psi} = \overset{c}{\psi} + \omega^{ct} \times \psi \tag{53}$$

Substitute Eqs. (48) and (53) into Eq. (52) to obtain

$$\delta\overset{t}{\theta} = -\omega^{ti} \times \phi + \delta_t \omega^{ti} + \epsilon - (\overset{c}{\psi} + \omega^{ct} \times \psi) \tag{54}$$

Next, substitute the expression for $\overset{c}{\dot{\boldsymbol{\psi}}}$ given in Eq. (38c) into Eq. (54) to obtain

$$\delta\overset{t}{\dot{\boldsymbol{\theta}}} = -\boldsymbol{\omega}^{ti} \times \boldsymbol{\phi} + (\boldsymbol{\omega}^{tc} + \boldsymbol{\omega}^{ci}) \times \boldsymbol{\psi} + \delta_t\boldsymbol{\omega}^{ti} \tag{55}$$

which yields

$$\delta\overset{t}{\dot{\boldsymbol{\theta}}} = -\boldsymbol{\omega}^{ti} \times (\boldsymbol{\phi} - \boldsymbol{\psi}) + \delta_t\boldsymbol{\omega}^{ti} \tag{56}$$

Using Eq. (27), the last equation becomes

$$\delta\overset{t}{\dot{\boldsymbol{\theta}}} = -\boldsymbol{\omega}^{ti} \times \delta\boldsymbol{\theta} + \delta_t\boldsymbol{\omega}^{ti} \tag{57}$$

Equations (51) and (57) yield the following absolute error model:

$$\delta_t\overset{t}{\dot{\boldsymbol{R}}} = \delta_t\boldsymbol{U} - \delta_t\boldsymbol{\omega}^{ti} \times \boldsymbol{R} - \boldsymbol{\omega}^{ti} \times \delta_t\boldsymbol{R} \tag{58a}$$

$$\delta_t\overset{t}{\dot{\boldsymbol{U}}} = -\delta_t\boldsymbol{\omega}^{ti} \times \boldsymbol{U} - \boldsymbol{\omega}^{ti} \times \delta\boldsymbol{U} + \delta_t\boldsymbol{f} + \delta_t\boldsymbol{g}^m \tag{58b}$$

$$\delta\overset{t}{\dot{\boldsymbol{\theta}}} = -\boldsymbol{\omega}^{ti} \times \delta\boldsymbol{\theta} + \delta_t\boldsymbol{\omega}^{ti} \tag{58c}$$

Note that Eqs. (58a), (58b), and (58c) correspond, respectively, to Eqs. (52c), (52b), and (52a) of Ref. 5.

V. Conclusions

A general approach to INS error model development was presented in this paper. This approach shows that the existing error models can be derived using several perturbation rules and several choices of navigational variables, error variables, reference (nominal) coordinates, and coordinate systems for linearization. That is, the differences between the various known models stem from different choices made by the developer of the models. This was clearly shown by three examples of known error models. The approach presented in this work has three advantages; namely, it shows how one can easily develop the INS error model suited to one's needs; it shows that all error models of the class of inertial systems considered herein are equivalent; and finally, it puts known and future INS error models of such systems into the same framework.

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